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Statistical mechanics of anyons

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Abstract. We propose a new generalized statistical mechanics for anyons based on a simple ansatz that has the correct familiar limiting cases. We derive the anyon equation of state from the grand partition function. We develop the virial expansion and obtain *all* the virial coefficients. Furthermore, we demonstrate how thermodynamic quantities such as the molar specific heat can readily be calculated. We discuss the properties of semions and also demonstrate that ideal non-relativistic anyons do not exhibit finite temperature condensation.

1. Introduction

Planar physical systems, in two space and one time dimensions, display many peculiar and interesting quantum properties which have to do with the unusual structure of the rotation, Lorentz and Poincaré groups in $2+1$ dimensions. These features lead to richer possibilities for the quantum mechanics of angular momentum and, in particular, give rise to the existence of quantum states with angular momentum not quantized in the familiar half-integer units. These objects, named anyons [1–7], have begun to attract a great deal of attention. The anyon may be relevant to the explanation of a variety of interesting physical phenomena, such as the Aharanov–Bohm effect, fractional quantized Hall effect, high-temperature superconductivity and others.

In two dimensions, a particle can be rotated around another particle and such a transformation enables the two-particle wavefunction to develop a phase which is not well defined in dimensions higher than two and is uniquely defined only in two dimensions. Consequently, the spin may be arbitrary, and on account of the connection between spin and statistics, the statistics is arbitrary in two dimensions.

Anyons lead to a generalization of the usual definition of statistics and the spin-statistics theorem. In two space dimensions, the wavefunction of the n-particle system $\psi(q_1, \ldots, q_n)$ will satisfy

$$\psi(q_1, \ldots, q_j, \ldots, q_i, \ldots, q_n) = \mathrm{e}^{\mathrm{i}\pi\alpha}\,\psi(q_1, \ldots, q_i, \ldots, q_j, \ldots, q_n) \qquad (1.1)$$

where the statistics parameter α is an arbitrary real number of modulo integer and the q_i refer to the coordinates and other quantum numbers. Bosons and fermions correspond to $\alpha = 0$ and 1, respectively. The generalization to any statistics corresponds to $\alpha =$ any real number [2, 4, 5].

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The anyon is a particle which carries both a 'charge' e and a magnetic flux Φ, which possesses a spin $s = -e\Phi/2hc$, which is, in general, neither an integer nor a half integer. The Chern–Simons theory (CS) provides a natural realization of the anyon with fractional statistics. There is a remarkable and mysterious connection between anyons and two-dimensional conformal field theory, i.e. the algebra of the conformal group $SO(2, 1)$, in two dimensions, is a symmetry group of CS theory.

Anyonic theory is determined by an angle θ which denotes the strength of the fictitious flux in the magnetic gauge. When two identical anyons are exchanged, the wavefunction develops a phase $e^{i\theta}$. Anyonic theories specified by θ and $\theta + 2\pi$ are identical and we can, therefore, restrict the variable to the interval $-\pi < \theta \leqslant \pi$. Furthermore, theories with θ and $-\theta$ are related by parity or time reversal and so one could restrict the range of θ to between 0 and π. The values $\theta = 0$ and π correspond to the bosonic and fermionic cases, respectively. Alternatively, one could characterize anyon statistics by the parameter $\alpha = \theta/\pi$, where in two dimensions α can vary continuously between 0 and 1 and thus interpolate between Bose and Fermi statistics, i.e. $\alpha = 0$ yields the Bose–Einstein (BE) distribution and $\alpha = 1$ corresponds to the Fermi–Dirac (FD) distribution. The intermediate value $\alpha = \frac{1}{2}$ corresponds to the semion. In the so-called anyon gauge, the anyon Hamiltonian is identical to those for bosons and fermions but the wavefunction satisfies the 'twisted' boundary condition: $\psi(\theta + 2\pi) = e^{i\alpha\theta}\psi(\theta)$, with $0 \leqslant \alpha \leqslant 1$. The breaking of the parity P and time reversal T discrete symmetries is considered the hall mark of anyon statistics and this provides a possible signal for experiments to detect the presence of anyons in two-dimensional phenomena.

The collective excitations above the ground state of systems which exhibit the fractional quantum Hall effect have been identified as quasi particles of fractional spin and charge which obey fractional anyon statistics. Anyons have also been conjectured to play an important role in the theory of high-temperature superconductivity. Furthermore, semions have also been conjectured to play a role in the pairing theory of high-temperature superconductors. It has been discovered by Laughlin [8] that an ideal anyon gas is a superfluid and, consequently, a gas of charged anyons would exhibit superconductivity.

The precise form of the statistical distribution function for anyons remains unknown and is the outstanding problem in anyon theory today. There have been interesting conjectures on the statistics interpolating between BE and FD statistics [9] but no clear answers have emerged. There have been perturbative investigations of the anyon gas [10]. This approach to the problem involves a perturbative study into the properties of the anyon gas at values close to $\theta = 0$ or π, regarded as an expansion around the boson or fermion, which can yield approximate results for the statistical properties of the anyon gas [11]. The exact two-particle partition function has been solved by Arovas *et al* [12] leading to a computation of the second virial coefficient in the virial expansion, which is valid for low densities and/or high temperatures. Some progress has been made in the computation of the third virial coefficient. Since determining the grand partition function and establishing the equation of state is a many-anyon problem, it has been regarded as an insoluble problem and no exact results are known.

It is the purpose of this paper to attack the problem in an entirely new fashion. We begin with the simplest natural generalization of the familiar BE and FD statistical distribution function. We are then lead to an ansatz which, to the best of our knowledge, correctly describes the statistical distribution for anyons. On the basis of this, we are able to develop a full statistical anyon theory and obtain all the usual statistical and thermodynamic quantities characterizing anyons. We hope this will lead to a better understanding of the statistical, thermodynamical and other properties of the anyon gas.

2. Equation of state and virial expansion: B-anyons

We begin with the requirement that the statistical distribution function for anyons should involve the statistics determining parameter α explicitly and it should have the correct smooth limiting forms when $\alpha = 0$ (bosonic case) and 1 (fermionic case). We, therefore, propose that the following ansatz provides the appropriate distribution function for the anyons:

$$N = \gamma \sum_k \frac{1}{[e^{\beta(E_k-\mu)} - f(\alpha)]} \tag{2.1}$$

where $f(\alpha)$ is a function of α which satisfies the boundary conditions or property: $f(0) = 1$, $f(+1) = -1$. We further require that $f(0) = 0$ in order to include the quantum Boltzmann case [13]. We shall also assume that the statistics determining parameter α and, hence, $f(\alpha)$ is independent of temperature. This ansatz is our conjecture and fundamental postulate and, aside from these boundary conditions, there are no restrictions on the function $f(\alpha)$. Here, k labels the energy states, $\beta = 1/kT$, μ is the chemical potential and $\gamma = \alpha + 1 = 2s + 1$ denotes the multiplicity. This form interpolates between BE statistics, with $\alpha = 0$, and FD statistics with $\alpha = 1$. The sum in equation (2.1) gives the correct distribution function for the cases of BE and FD statistics but there is a problem with the case of anyons since the N-body wavefunction cannot be constructed from products of single-particle wavefunctions, because one has to represent the braid group. Therefore, one needs to work in the anyon gauge [5] when the Hamiltonian is identical for bosonic, fermionic and anyonic systems but the many-body wavefunctions have different symmetry properties for different statistics.

There are sound reasons why we should consider the two cases of positive and negative values for $f(\alpha)$ separately. For the case of negative $f(\alpha)$, let us set $f(\alpha) = -g(\alpha)$, and, thus, consider the generalization which enables us to deal with two types of anyon. The range of α which corresponds to $f > 0$ and $g > 0$ will not be specified at this point. Indeed, we shall determine and discuss it later. Accordingly, the following distribution functions describe boson-like (B-anyons, abbreviated as BA) and fermion-like anyons (F-anyons, abbreviated as FA):

$$N^{\mathrm{BA}} = \gamma \sum_k \frac{1}{e^{\beta(E_k-\mu)} - f} \qquad N^{\mathrm{FA}} = \gamma \sum_k \frac{1}{e^{\beta(E_k-\mu)} + g} \tag{2.2}$$

henceforward, regarding f and g as functions of the parameter α. We shall first investigate the equation of state for boson-like anyons, without concerning ourselves with condensation. Thus, the average occupation number of B-anyons is given by

$$N^{\mathrm{BA}} = \gamma \sum_E \frac{1}{e^{\beta(E-\mu)} - f} \tag{2.3}$$

where $e^{\beta\mu} = z$ is the fugacity. Corresponding to this ansatz for the anyons, we propose the following ansatz for the grand partition function, analogous to the bosonic case

$$\mathcal{Z}^{\mathrm{BA}} = \frac{1}{f} \prod_{l=0}^{\infty} \left(\frac{1}{1 - f e^{-\beta(E_l-\mu)}} \right)^{\gamma}. \tag{2.4}$$

This is an extension of the familiar expression for BE statistics, with the introduction of $f(\alpha)$ in the anyon gauge. This expression takes account of the multiplicity factor

$\gamma = 2s + 1 = \alpha + 1$ for the reason that α denotes the 'spin' of the anyon. The interpolating statistics for the anyons also corresponds to an interpolating or variable spin [2]. The limit $\alpha = 0$ yields the usual BE result [4]:

$$\mathcal{Z}^{\mathrm{B}} = \prod_{i=0}^{\infty} \frac{1}{1 - \mathrm{e}^{-\beta(E_i - \mu)}}. \tag{2.5}$$

We find the grand potential to be

$$\Omega^{\mathrm{BA}} = -kT \ln \mathcal{Z}^{\mathrm{BA}} = \frac{1}{f} kT\gamma \sum \ln(1 - fz\mathrm{e}^{-\beta E}). \tag{2.6}$$

We can compute the average occupation number $N = -\partial\Omega/\partial\mu$ and obtain

$$N = -\beta z \frac{\partial \Omega}{\partial z} = \gamma \sum_i \frac{1}{\mathrm{e}^{\beta(E_i - \mu)} - f}. \tag{2.7}$$

This is consistent with the original ansatz proposed and, hence, the definition in equation (2.4) correctly describes B-anyons. We proceed to compute the grand potential as follows. If we isolate the contribution from the zero-momentum state, we find

$$\Omega_{\mathrm{BA}} = \frac{kT\gamma}{f} \ln(1 - fz) + \frac{kT\gamma}{f} \sum_i \ln(1 - fz\mathrm{e}^{-\beta E_i}). \tag{2.8}$$

By converting the sum to an integration, the second term of equation (2.8) can be expressed as a one-dimensional integral:

$$\frac{VkT\gamma}{\lambda^2} \int_0^{\infty} \mathrm{d}x \, \ln[1 - fz\mathrm{e}^{-x}]. \tag{2.9}$$

Here, $\lambda^2 = 2\pi\beta\hbar^2/m$ where λ is the thermal wavelength and V is the two-dimensional volume. Employing a power series expansion and integrating term by term, we can express this second term as

$$-\frac{VkT\gamma}{f\lambda^2} F[fz] \tag{2.10}$$

where the function F is defined by

$$F[z] = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \tag{2.11}$$

This is a sort of generalization of the familiar Riemann zeta function $\zeta(2)$. This function possesses the following integral transform [15]:

$$F[z] = -\int_0^z \frac{\ln(1-u)}{u} \, \mathrm{d}u \tag{2.12}$$

and obeys the property

$$F'[z] = -\frac{\ln(1-z)}{z} \tag{2.13}$$

where the prime denotes the derivative with respect to the argument. We, therefore, obtain

$$\frac{\Omega^{\mathrm{BA}}}{V} = -\frac{kT\gamma}{V}\ln(1-fz) - \frac{kT\gamma}{f\lambda^2}F[fz]. \tag{2.14}$$

Upon evaluating the thermodynamic limit $V \to \infty$, $N \to \infty$, we obtain the expression for the pressure of the B-anyon gas:

$$P^{\mathrm{BA}} = \lim_{V\to\infty, N\to\infty}\left(-\frac{\Omega}{V}\right) = \frac{kT\gamma}{f\lambda^2}F[fz]. \tag{2.15}$$

We can derive a similar result for the occupation number. The summation in equation (2.3) can be converted into an integration, thus, expressing the average occupation number as a one-dimensional integral

$$N^{\mathrm{BA}} = -\frac{V\gamma}{f\lambda^2}\int_0^\infty \frac{\mathrm{d}x}{z^{-1}\mathrm{e}^x - f}. \tag{2.16}$$

Employing a power series and integrating, we obtain

$$N^{\mathrm{BA}} = -\frac{V\gamma}{f\lambda^2}\ln(1-fz). \tag{2.17}$$

By combining the above two relations, we obtain the important result

$$\left(\frac{PV}{NkT}\right)_{\mathrm{BA}} = -F[fz]\frac{1}{\ln(1-fz)}. \tag{2.18}$$

This is the equation of state for the ideal B-anyon gas derived in the thermodynamic limit. Evidently, it is not in a closed form. The parameter $|f|$ is smaller than unity for anyons. The parameter z is temperature dependent and is small in the high-temperature–low-density approximation and this, therefore, assures a rapid convergence if we obtain the virial expansion from an expansion in powers of z. For this purpose, let us first expand the logarithm. The equation of state is thus

$$\left(\frac{PV}{NkT}\right)_{\mathrm{BA}} = -F[fz]\left\{fz + \frac{(fz)^2}{2} + \frac{(fz)^3}{3} + \cdots\right\}^{-1} \tag{2.19}$$

where the function F is given by a power-series expansion. Next, we need to develop a power-series expansion for z, for which we utilize the result in equation (2.17). This yields

$$1 - fz = 1 - \mathrm{e}^{-f\lambda^2 N/\gamma V} \tag{2.20}$$

and we, therefore, determine z as a function of the density $\rho = N/V$

$$z = \frac{P1}{f}(1 - \mathrm{e}^{-f\lambda^2\rho/\gamma}) \tag{2.21}$$

which has the power-series expansion

$$z = \frac{\lambda^2}{\gamma}\rho\left[1 - \frac{1}{2!}f\lambda^2\rho/\gamma + \frac{1}{3!}(f\lambda^2\rho/\gamma)^2 - \cdots\right]. \tag{2.22}$$

This can be inserted into the equation of state (2.19). We now perform the necessary symbolic mathematics on MATHEMATICA [16] and obtain the complete virial expansion. We recall the standard form for the virial expansion:

$$\left(\frac{PV}{NkT}\right)_{\mathrm{BA}} = B_1 + B_2\left(\frac{\lambda^2 N}{\gamma V}\right)^2 + B_3\left(\frac{\lambda^2 N}{\gamma V}\right)^3 + \cdots. \tag{2.23}$$

For the case of B-anyons, we adhere to the notation convention used for the BE case [14, 17].

Using the notation $x = \lambda^2\rho/\gamma = \lambda^2 N/\gamma V$, we obtain the virial coefficients, in terms of f, from the series

$$\left(\frac{PV}{NkT}\right)_{\mathrm{BA}} = B_1 + B_2 x + B_3 x^2 + \cdots + B_{10}x^9 + \mathrm{O}(x^{11}). \tag{2.24}$$

The first few coefficients are listed below:

$$B_1 = 1 \tag{2.25}$$

$$B_2 = -\tfrac{1}{4}f \tag{2.26}$$

$$B_3 = \tfrac{1}{36}f^2. \tag{2.27}$$

The virial coefficients up to order eleven are listed in the appendix. We could list the virial coefficients to any desired order but we did not see any point in going beyond the eleventh. It is important to stress that in view of the nature of the symbolic mathematics performed on the computer by MATHEMATICA, *all these results for the virial expansion are exact, as are all the results in this paper, with no approximations made whatsoever.*

3. Equation of state and virial expansion: F-anyons

Let us begin with the occupation number for the F-anyons introduced earlier

$$N^{\mathrm{FA}} = \gamma \sum_k \frac{1}{\mathrm{e}^{\beta(E_k-\mu)} + g(\alpha)}. \tag{3.1}$$

Analogous to the case of B-anyons, we find it convenient to regard $g(\alpha)$ as a function of the standard statistics parameter α. To this number distribution corresponds the following grand partition function for the fermion-like anyons:

$$\mathcal{Z}^{\mathrm{FA}} = \frac{1}{g}\prod_{i=0}^{\infty}[1 + g\mathrm{e}^{-\beta(E_i-\mu)}]^{\gamma} = \frac{1}{g}\prod_{i=0}^{\infty}[1 + gz\mathrm{e}^{-\beta E_i}]^{\gamma}. \tag{3.2}$$

This form conforms to the Fermi case, with the inclusion of the statistics parameter $g(\alpha)$, and reduces the familiar FD grand partition function [17] in the limit when $\alpha = 1$, $g(\alpha) = 1$. The exponent is due to the multiplicity factor $\gamma = 2s + 1 = \alpha + 1$. This is due to the fact that the anyons which obey an interpolating statistics also have interpolating spin given by $s = \alpha/2$. We find the grand potential to be

$$\Omega = -kT\ln\mathcal{Z} = -\frac{1}{g}kT\gamma\sum\ln(1 + gz\mathrm{e}^{-\beta E_i}). \tag{3.3}$$

We determine the average occupation number to be

$$N^{\mathrm{FA}} = -\frac{\partial\Omega}{\partial\mu} = -\beta z\frac{\partial\Omega}{\partial z} = \gamma\sum\frac{1}{e^{\beta(E_i-\mu)}+g} \tag{3.4}$$

and, thus, the grand partition function of equation (3.2) correctly reproduces the ansatz for the occupation number for F-anyons. If we convert the sum to an integral, equation (3.4) can be expressed as

$$N^{\mathrm{FA}} = \frac{\gamma V}{\lambda^2}\int_0^\infty \frac{\mathrm{d}x}{z^{-1}e^x+g}. \tag{3.5}$$

Integration by series expansion yields

$$N^{\mathrm{FA}} = \frac{\gamma V}{g\lambda^2}\ln(1+gz). \tag{3.6}$$

We compute the grand potential as follows. Isolating the zero-momentum state, we obtain

$$\Omega = -\frac{\gamma kT}{g}\ln(1+gz) - \frac{\gamma kT}{g}\sum\ln(1+gze^{-\beta E_i}). \tag{3.7}$$

If we replace the sum by an integral, the second term on the right-hand side may be expressed as

$$-\frac{\gamma kTV}{g\lambda^2}\int_0^\infty \mathrm{d}x\,\ln(1+ge^{-x}). \tag{3.8}$$

Employing a series expansion, we, thus, obtain

$$\frac{\Omega}{V} = -\frac{\gamma kT}{gV}\ln(1+gz) - \frac{\gamma kT}{g\lambda^2}G[zg] \tag{3.9}$$

where the function G occurring here is defined by

$$G[z] = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^2}. \tag{3.10}$$

We note that the function G defined above has the following integral transform:

$$G[z] = \int_0^z \frac{\ln(1+u)}{u}\,\mathrm{d}u \tag{3.11}$$

and satisfies the property

$$G'[z] = \frac{\ln(1+z)}{z} \tag{3.12}$$

where the prime denotes the derivative with respect to the argument. The pressure of the F-anyon gas is determined by evaluating the thermodynamic limit $V \to \infty$, $N \to \infty$:

$$P^{\mathrm{FA}} = \lim_{V\to\infty,N\to\infty}\left(-\frac{\Omega}{V}\right) = P^{\mathrm{FA}} = \frac{\gamma kT}{g\lambda^2}G[zg] \tag{3.13}$$

a result different from the case of B-anyons. We can now proceed to develop the virial expansion for the F-anyon gas. If we combine the above result with the distribution function given by equation (3.6), we obtain the result

$$\left(\frac{PV}{NkT}\right)_{\mathrm{FA}} = G[zg]\frac{1}{\ln(1+zg)}. \qquad (3.14)$$

For high temperatures, $\beta\mu$ is large and negative and $z \sim 0$, whereas for low temperatures, $\beta\mu$ is large and positive and $z \sim \infty$. The virial expansion is a high-temperature and/or low-density expansion. Expanding the logarithm in a series which converges for $z^2g^2(q) < 1$ and $zg(q) = 1$, we obtain

$$\left(\frac{PV}{NkT}\right)_{\mathrm{FA}} = G[zg]\left\{zg - \frac{(zg)^2}{2} + \frac{(zg^3)}{3} - \cdots\right\}^{-1}. \qquad (3.15)$$

This leads to the following power-series expansion to express z in terms of the density, which follows from equation (3.6):

$$z = \frac{1}{g}(\mathrm{e}^{g\rho\lambda^2/\gamma} - 1) = \lambda^2\rho/\gamma + \frac{1}{2!}g(\lambda^2\rho/\gamma)^2 + \cdots. \qquad (3.16)$$

We can then develop a power series in the variable z, by using MATHEMATICA to perform the necessary symbolic mathematics. We can read off the virial coefficients by comparing the output from MATHEMATICA with the standard virial expansion for fermion-like objects in the standard notation [14]:

$$\left(\frac{PV}{NkT}\right)_{\mathrm{FA}} = B_1 - B_2\left(\frac{\lambda^2 N}{\gamma V}\right) + B_3\left(\frac{\lambda^2 N}{\gamma N}\right)^2 + \cdots. \qquad (3.17)$$

The virial coefficients for the F-anyons turn out to be exactly the same as the ones listed in the appendix with f replaced by g. This circumstance is reminiscent of the virial coefficients for the ordinary BE and FD systems in three dimensions.

4. Analysis of the statistics parameters

First, in order to relate our parameters $f(\alpha)$ and $g(\alpha)$ to the standard statistics parameter $\alpha = \theta/\pi$, we proceed as follows. Arovas *et al* [12] have investigated the problem of the two-dimensional free-anyon gas and have derived the following exact result for the second virial coefficient of B-anyons by employing an expansion about the Bose point $\theta = 0$:

$$B_2^{\mathrm{BA}} = -\tfrac{1}{4} + \alpha - \tfrac{1}{2}\alpha^2. \qquad (4.1)$$

We shall use this result as a means of calibrating our results for the virial coefficients of B-anyons. The result in [12] is expressed in terms of another parameter δ defined by $\alpha = n+\delta$ but we can set $n = 0$ with no loss of generality. If we require our results to conform to this exact result, then we have, for B-anyons,

$$B_2^{\mathrm{BA}} = -\tfrac{1}{4} + \alpha - \tfrac{1}{2}\alpha^2 = -\tfrac{1}{4}f. \qquad (4.2)$$

This determines our statistics parameter f in terms of the standard parameter α:

$$f(\alpha) = 1 - 4\alpha + 2\alpha^2. \tag{4.3}$$

Analogously, for F-anyons, we obtain the calibration, using Arovas' result for the second virial coefficient, by employing an expansion about the Fermi point $\theta = \pi$:

$$B_2^{\mathrm{FA}} = \tfrac{1}{4} - \tfrac{1}{2}\alpha^2 = -\tfrac{1}{4}g. \tag{4.4}$$

We can solve for g and, thus, arrive at the determination

$$g(\alpha) = 2\alpha^2 - 1. \tag{4.5}$$

It is interesting to observe that the virial coefficients for the B-anyons and F-anyons have the same form in terms of the parameter f or g but f and g are not the same in terms of α. Hence, expressed in terms of the statistics parameter α, the virial coefficients are different for BA and FA.

The boson, fermion and semion limits are then as follows

$$\alpha = 0 \to f = 1, g = -1 \tag{4.6}$$

$$\alpha = 1 \to f = -1, g = 1 \tag{4.7}$$

$$\alpha = \tfrac{1}{2} \to f = -\tfrac{1}{2}, g = -\tfrac{1}{2}. \tag{4.8}$$

It is gratifying to observe that the boson and fermion limits come out exactly right. Moreover, it is interesting that even though there are two parameters f and g, which coincide at $\alpha = \frac{1}{2}$, it turns out that the semion is not unique. Indeed there are two kinds of semions, which will be discussed in the next section.

The quantum Boltzmann case of $f = 0$ corresponds to $\alpha = 1 - 1/\sqrt{2}$ and the case of $g = 0$ corresponds to $\alpha = 1/\sqrt{2}$. The case of $f = 0$ is interesting and deserves examination. Much of the analysis of sections 3 and 4 is performed after division by f or g. Therefore to study this case, we must return to the original expressions for the total number of particles and energy. Carrying out the integration in the B-anyon case, for instance, we obtain

$$(E)_{f=0} = \frac{\gamma VkT}{\lambda^2}\mathrm{e}^{\beta\mu} \tag{4.9}$$

and

$$(N)_{f=0} = \frac{\gamma V}{\lambda^2}\mathrm{e}^{\beta\mu}. \tag{4.10}$$

This leads to

$$E = NkT \qquad PV = E = NkT \qquad B_1 = 1 \qquad B_n(n \neq 0) = 0 \tag{4.11}$$

as expected.

Considered as a function of the parameter α, the parameter functions are seen to exhibit Bose–Fermi (B–F) symmetry. The function f, when expressed in terms of α, has the property

$$f(1-\alpha) = g(\alpha) \qquad g(1-\alpha) = f(\alpha). \tag{4.12}$$

Thus $f \to g$ when $\alpha \to 1-\alpha$. Consequently, we note that all the virial coefficients possess a symmetry: $B_n^{\rm FA}(1-\alpha) = B_n^{\rm BA}(\alpha)$. However, it is not true that the $B_n^{\rm BA}$ possess a kind of mirror symmetry, which would equate $B^{\rm BA}(\alpha)$ and $B^{\rm BA}(1-\alpha)$. For instance, we can calculate $B_3^{\rm BA}(\alpha) - B_3^{\rm BA}(1-\alpha)$ and find

$$B_3^{\rm BA}(\alpha) - B_3^{\rm BA}(1-\alpha) = \tfrac{4}{9}\alpha(\alpha-1)(1-2\alpha). \tag{4.13}$$

We see that this vanishes only for $\alpha = 0, 1, \frac{1}{2}$, which are precisely the bosonic, fermionic and semionic cases. Such a symmetry does not prevail, in general, for the anyons†.

We observe from the virial coefficients listed in the appendix that all the even coefficients beyond the second coefficient vanish: $B_{2n} = 0$, $n > 2$. This remarkable result is reminiscent of the standard bosonic or fermionic case. Finally, in the limit $f = 1$, the coefficients we have obtained revert to the exact values known for bosons [18].

5. Semions

The semionic case [19] corresponding to $\alpha = \frac{1}{2}$ is interesting. Setting $\alpha = \frac{1}{2}$, we find the results

$$f(\tfrac{1}{2}) = -\tfrac{1}{2} \qquad g(\tfrac{1}{2}) = -\tfrac{1}{2}. \tag{5.1}$$

Semionic statistical mechanics is determined by

$$N^{\rm Sem} = \tfrac{3}{2}\sum \frac{1}{(e^{\beta(E-\mu)} \pm \frac{1}{2})}. \tag{5.2}$$

Consequently, there exist two kinds of semion; in a sense, the middle as approached from the Bose point and the Fermi point. We do not know the implications of this result and are not aware if this has previously been conjectured. It is conceivable that this may be a reflection of the non-trivial topology in anyon space.

6. Condensation

Do ideal anyons exhibit a condensation phase at finite temperature? It is important to investigate this question in view of the conjecture by Laughlin [8] that there is an anyon condensation phenomenon. Let us examine

$$N_p = \frac{\gamma}{e^{\beta(E-\mu)} - f(\alpha)} = \frac{\gamma}{z^{-1}e^{\beta E} - f(\alpha)} \tag{6.1}$$

where $z = e^{\beta\mu}$ and $0 < f < 1$. The zero-momentum state can be isolated as

$$N_0 = \gamma \frac{z}{1 - f(q)z}, \tag{6.2}$$

† The strong result quoted in [5], i.e. $B_3^{\rm BA}(\alpha) = B_3^{\rm BA}(1-\alpha)$ is perhaps an artefact of the properties of the supersymmetric operator Q. This operator Q ceases to exist in the limit $\omega \to 0$ and, hence, cannot be used to draw any conclusion in that limit.

The denominator in N_p must be non-negative and, thus, the range for z is given by $0 \leqslant z \leqslant 1/f$ and we find that N_0 becomes macroscopically large when $z = 1/f$ for $0 < \alpha < 1 - 1/\sqrt{2}$. In the limit when $z = 1/f$, we find $\beta\mu = -\ln f$, where $0 < f < 1$. Isolating the zero-momentum state, we can calculate N/V after replacing the sum by an integration and thus

$$\frac{N}{V} = \frac{N_0}{V} - \gamma \frac{1}{f\lambda^2} \ln(1 - fz) \tag{6.3}$$

$$= \frac{N_0}{V} + \gamma \frac{1}{f\lambda^2} \sum_{n=1}^{\infty} \frac{[fz]^n}{n}. \tag{6.4}$$

For low densities and/or high temperatures, N_0 is negligible, N/V is essentially given by the second expression and there is nothing more to be said. Let us consider the limiting case of $z = 1/f$. The series in equation (6.4) will converge only for $|fz| < 1$ and $fz = -1$, whereas condensation requires $fz = 1$. Hence, anyons do not appear to exhibit condensation at a finite temperature. Thus, the non-condensation result for bosons in two dimensions also applies to anyons.

In a similar manner, we can investigate the case of F-anyons, starting with the expression for the occupation number

$$N = \sum N_p = \sum \frac{1}{z^{-1}e^{\beta E} + g(\alpha)} \tag{6.5}$$

where g is restricted to $-1 > g(\alpha) > 0$. The denominator must remain non-negative for the unrestricted fugacity range $0 < z < \infty$. Analogously to equation (6.3), this gives rise to a term containing $\ln(1 + gz)$ and the series does not converge for $gz = -1$, which is when condensation can take place. Hence F-anyons do not exhibit condensation.

7. The specific heat of anyons

We begin with the distribution functions for the two types of anyon. Introducing the density of states and replacing the sum by an integration, we can express various quantities as one-dimensional integrals. Invoking the equipartition principle, we have the following expressions for the energy of the B- and F-anyons:

$$E^{\mathrm{BA}} = \frac{kTV\gamma}{\lambda^2} \int_0^{\infty} \mathrm{d}x \frac{1}{e^{x-\beta\mu} - f} \tag{7.1}$$

and

$$E^{\mathrm{FA}} = \frac{kTV\gamma}{\lambda^2} \int_0^{\infty} \mathrm{d}x \frac{1}{e^{x-\beta\mu} + g}. \tag{7.2}$$

Employing a power series expansion and integrating term by term, we have

$$N^{\mathrm{BA}} = -\frac{V\gamma}{\lambda^2} \frac{1}{f} \ln(1 - fe^{\beta\mu}) \tag{7.3}$$

and

$$N^{\mathrm{FA}} = \frac{V\gamma}{\lambda^2} \frac{1}{g} \ln(1 + ge^{\beta\mu}). \tag{7.4}$$

We can compute the energy of anyons in a similar manner. We obtain

$$E^{\mathrm{BA}} = \frac{kTV\gamma}{\lambda^2}\frac{1}{f}F[f\mathrm{e}^{\beta\mu}] \tag{7.5}$$

and

$$E^{\mathrm{FA}} = \frac{kTV\gamma}{\lambda^2}\frac{1}{g}G[g\mathrm{e}^{\beta\mu}]. \tag{7.6}$$

All this analysis is reminiscent of the quantum statistics of two-dimensional gases carried out by May [15]. Accordingly, we introduce the temperature T_0 which divides the classical regime ($T > T_0$) from that of quantum statistics ($T < T_0$). We further employ the parameters

$$N = V/\lambda_0^2 \qquad \tau_f = \mathrm{e}^{-fT_0/T} \qquad \tau_g = \mathrm{e}^{-gT_0/T} \tag{7.7}$$

and note that

$$\lambda^2/\lambda_0^2 = T_0/T. \tag{7.8}$$

Accordingly, we find that the chemical potential, for the two cases determined by

$$(\mathrm{e}^{\beta\mu})_{\mathrm{BA}} = \frac{1}{f}(1 - \mathrm{e}^{-fT_0/T}) = \frac{1}{f}(1 - \tau_f) \tag{7.9}$$

and

$$(\mathrm{e}^{\beta\mu})_{\mathrm{FA}} = \frac{1}{g}(\mathrm{e}^{gT_0/T} - 1) = \frac{1}{g}\left(\frac{1}{\tau_g} - 1\right). \tag{7.10}$$

The energy of the anyons is then given by

$$E^{\mathrm{BA}} = \frac{kTV\gamma}{\lambda^2}\frac{1}{f}F[f\mathrm{e}^{\beta\mu}] = \frac{\gamma NkT^2}{T_0}\frac{1}{f}F[1 - \tau_f] \tag{7.11}$$

and

$$E^{\mathrm{FA}} = \frac{kTV\gamma}{\lambda^2}\frac{1}{g}G[g\mathrm{e}^{\beta\mu}] = \frac{\gamma NkT^2}{T_0}\frac{1}{g}G\left[\frac{1}{\tau_g} - 1\right]. \tag{7.12}$$

We can now compute the specific heat

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V}. \tag{7.13}$$

We obtain, for the boson- and fermion-like anyons,

$$\hat{C}_V^{\mathrm{BA}} = \frac{2NkT}{T_0}\frac{1}{f(\alpha)}F[1 - \tau_f] + Nk\tau_f\frac{\ln\tau_f}{1 - \tau_f} \tag{7.14}$$

and

$$\hat{C}_V^{\rm FA} = \frac{2NkT}{T_0}\frac{1}{g(\alpha)}G\left[\frac{1}{\tau_g} - 1\right] + Nk\frac{\ln\tau_g}{1-\tau_g}. \tag{7.15}$$

Here we have introduced the notation $\hat{C} = C/\gamma$. We see that in the usual bosonic and fermionic limit, this reduces to the familiar result [15], namely, that the two specific heats modulo the multiplicity factor, in two dimensions, become identical

$$\hat{C}_V^{\rm B} = \hat{C}_V^{\rm F}. \tag{7.16}$$

We may recall that [15] ignored the multiplicity factor γ. The equality of the specific heats is true only in the above form. We can also study the high-temperature limit as follows. In the classical regime, for $T > T_0$, we have

$$\tau_f \approx 1 - fT_0/T \qquad F[1-\tau_f] \approx 0 \tag{7.17}$$

and

$$\tau_g \approx 1 - gT_0/T \qquad G[1-\tau_g] \approx 0. \tag{7.18}$$

In this case, we obtain the expected classical result for an ideal gas in two dimensions at high temperatures:

$$C_V^{\rm BA} \approx C_V^{\rm FA} \approx Nk. \tag{7.19}$$

Finally, we can demonstrate that the results for the specific heats contained in equations (7.14) and (7.15) vanish in the limit $T \to 0$ as required by the third law of thermodynamics. (The vanishing of entropy at zero temperature requires the vanishing of specific heat.) For this purpose, we need the following property:

$$G\left[\frac{1}{z} - 1\right] = F[1-z] + \tfrac{1}{2}(\ln z)^2 \tag{7.20}$$

which follows from equations (2.12) and (3.11). We find $\tau_f \to 0$ as $T \to 0$ and $\ln\tau_f = -fT_0/T$ diverges in that limit. Thus, we obtain the limit of the specific heat of B-anyons:

$$\lim_{T\to 0}\hat{C}_V^{\rm BA} = \lim_{T\to 0}\frac{2NkT}{T_0}\frac{1}{f(\alpha)}F[1] + \lim_{T\to 0} Nk\tau_f\ln\tau_f. \tag{7.21}$$

We note that $F[1] = \zeta(2) = \pi^2/6$. An inspection shows that each term in the right-hand side above vanishes for arbitrary values of f as the temperature approaches zero. For the case of F-anyons, we find

$$\lim_{T\to 0}\hat{C}_V^{\rm FA} = \lim_{T\to 0}\frac{2NkT}{T_0}\frac{1}{g(\alpha)}\{F[1-\tau_g] + \tfrac{1}{2}(\ln\tau_g)^2\} + \lim_{T\to 0} Nk\ln\tau_g. \tag{7.22}$$

The first term on the right-hand side vanishes and the other two terms cancel and, hence, the specific heat of the F-anyons vanishes as the temperature approaches zero.

8. Specific heat and B–F symmetry

In the notation of the current literature, anyon statistics is controlled by the statistics parameter $\alpha = \theta/\pi$: the BE distribution corresponds to $\alpha = 0$ and the FD statistical distribution to $\alpha = 1$. There exists a symmetry, which we may call B–F symmetry which corresponds to the transformation $\alpha \to 1-\alpha$, sometimes referred to as the mirror symmetry. However, this transformation should not be allowed to affect the multiplicity factor γ, occurring in the expressions for specific heat, i.e. we should consider the transformation $\alpha \to 1-\alpha$, strictly modulo this factor. Due to the symmetry under this transformation, we find the following property:

$$g(1-\alpha) = f(\alpha) \tag{8.1}$$

which implies the transformation FA $\to$ BA. Accordingly, we expect various physical quantities corresponding to an F-anyon to approach the same quantities corresponding to a B-anyon under the replacement $\alpha \to 1-\alpha$. We may now use the identity stated in equation (7.20) to obtain the result

$$\hat{C}_V^{\mathrm{FA}}(1-\alpha) = \hat{C}_V^{\mathrm{BA}}(\alpha) \tag{8.2}$$

which is an anyon generalization of May's result [15]. Thus, the specific heat possesses the B–F symmetry.

We can now determine the specific heat of semions from the distribution function already established for the two types of semion. Setting $f = -\frac{1}{2}$ and $g = -\frac{1}{2}$ in the general distribution function for the average occupation number, we find that the B-anyons and F-anyons have switched roles, i.e. the F-semion corresponds to setting $f = \frac{1}{2}$ in equation (7.14) and the B-semion corresponds to setting $g = \frac{1}{2}$ in equation (7.15), respectively. Setting $\alpha = \frac{1}{2}$ and carrying out the calculation, we obtain the following results (recall that $C_V = \gamma \hat{C}_V$):

$$C_V^{\text{F-Sem}} = \frac{6NkT}{T_0} G\left[\frac{1}{z_0} - 1\right] - 3Nk\frac{T_0}{4T(1-z_0)} \tag{8.3}$$

and

$$C_V^{\text{B-Sem}} = \frac{6NkT}{T_0} F[1-z_0] - 3Nk\frac{T_0}{4T}\frac{z_0}{1-z_0} \tag{8.4}$$

where

$$z_0 = \mathrm{e}^{-T_0/2T}. \tag{8.5}$$

Finally, now using the identity (7.20), we find that the above two equations become identical, despite appearances, and, thus,

$$C_V^{\text{B-sem}} = C_V^{\text{F-sem}}. \tag{8.6}$$

9. Summary and conclusion

The exact statistical distribution function for the anyons has been an outstanding problem in anyon theory. In this paper, we have attacked this problem directly and in an entirely new fashion by proposing an ansatz for the explicit anyon distribution function. We have not derived it from any axiomatic basis and, hence, what we have proposed is a conjecture. However, it appears to be a natural and simple extension of the usual statistical distribution function. In this connection, it is interesting to note Wilczek's statement [2] that 'it may be well to state explicitly that the issues involved in defining particle quantum statistics are very different from those involved in formulating the commutation or anticommutation relations for local quantum fields. Whether the additional possibilities for quantum statistics of particles can be used to define additional possibilities for field quantization in any direct way, I do not know.' We are currently investigating this question.

Starting from the basic ansatz for the statistical distribution function, and consequently the grand partition function for anyons, we have developed the equation of state, the virial expansion and obtained exact analytical answers for the virial coefficients. We have shown how physical quantities such as the specific heat can be calculated. The expressions we obtain possess the correct low- and high-temperature limits. We have also demonstrated that all our expressions and the virial coefficients approach the familiar bosonic and fermionic cases in the appropriate limits.

In our investigation, we found that it is best to deal with boson-like and fermion-like anyons separately. We find the virial coefficients are the same or different depending on which parameter (f, g or α) is used for the bosonic and fermionic anyons, but there are other properties which are also different. When expressed in terms of the statistics determining parameter $\alpha = \theta/\pi$, we find that the statistical distribution functions are clearly different for the two types of anyon. We have shown how our parameters f and g, that we have introduced, can be related to the standard parameter α and determine f and g as functions of the parameter α. Thus, $f = 1 - 4\alpha + 2\alpha^2$ and $g = 2\alpha^2 - 1$. Our investigations also lead to some understanding of the statistical mechanics of semions.

We have demonstrated the absence of condensation at finite temperature for an ideal gas of non-relativistic anyons. The case of relativistic anyons is interesting [20] and is under investigation.

We have also briefly discussed the case of quantum Boltzmann statistics corresponding to $f = 0$, $\alpha = 1 - 1/\sqrt{2}$ and $g = 0$, $\alpha = 1/\sqrt{2}$. It is interesting to note that this implies two kinds of anyon, both obeying the same quantum Boltzmann statistics. In both cases, despite the vanishing of all the virial coefficients, there is a non-zero Aharanov–Bohm [2] scattering

$$\frac{d\sigma}{d\phi} = \frac{1}{2\pi k}\frac{\sin^2 \pi\alpha}{\sin^2 \phi/2}. \tag{9.1}$$

This effect, while zero for bosons and fermions, is non-zero for anyons in general and the quantum Boltzmann case in particular. This effect is what makes this limit $f = 0$ ($g = 0$) the *quantum* Boltzmann case.

It is interesting to speculate whether our approach to the anyon problem (of the distribution function) has any connection to q-mutators or q-oscillators. The subject of 'q-deformed statistical distributions' has been investigated by Parthasarathy and Viswanathan [21] who introduced such a distribution by considering the q-boson algebra [22]

$$AA^\dagger - qA^\dagger A = 1 \qquad [N, A] = -A \qquad [N, A^\dagger] = A^\dagger \tag{9.2}$$

with the number operator N given by

$$AA^{\dagger} - A^{\dagger}A = q^{N} \tag{9.3}$$

and, interestingly, not by $A^{\dagger}A$. Using a model for the Hamiltonian defined by

$$H = \sum_{k}(E_k - \mu)N_k \tag{9.4}$$

one then introduces the thermal-averaged occupation number $[N_k]$ which actually turns out to be a power series involving N_k, which can be interpreted as including interactions, such that it becomes N_k in the limit $q \to 1$. One then establishes that this occupation number is given by

$$\langle [N_k] \rangle = \frac{1}{e^{\beta(E_k-\mu)} - q} \tag{9.5}$$

A parallel analysis holds [21] for the q-deformed fermion case.

We are aware that anyons are not local objects and are representations of the braid group. Thus, anyons cannot be equivalent to q-mutators, q-deformed bosons and fermions or q-oscillators†. However, our analysis does indeed lead us to speculate that there may be some connection between anyons and 'generalized q-oscillators', with the generalization $q \to \phi(q)$ simulating nonlinear effects. This needs further investigation.

In this paper, we have developed a statistical mechanics for anyons. The ramifications for two-dimensional systems in the phenomenon of superfluidity, magnetism etc will be reported in a forthcoming publication.

Appendix. Virial coefficients of the anyon gas

$$B_1 = 1 \qquad B_2 = -\frac{1}{4}f \qquad B_3 = \frac{1}{36}f^2 \qquad B_4 = 0$$

$$B_5 = \frac{1}{3600}f^4 \qquad B_6 = 0 \qquad B_7 = \frac{1}{211\,680}f^6 \qquad B_8 = 0$$

$$B_9 = \frac{1}{10\,886\,400}f^8 \qquad B_{10} = 0 \qquad B_{11} = \frac{24\,129\,341}{2\,286\,144\,000}f^{10}.$$

The above are the virial coefficients for B-anyons. These can be expressed in terms of the parameter $\alpha = \theta/\pi$ by employing the result $f = 1 - 4\alpha + 2\alpha^2$. The boson limit corresponds to $f = 1$. Correspondingly, for F-anyons, the virial coefficients are the same as above, with the replacement $f \to g$. In that case, the substitution $g = 2\alpha^2 - 1$ will express the virial coefficients in terms of the standard parameter α. The fermion limit corresponds to $g = 1$. The semions correspond to $f = g = -\frac{1}{2}$ and the quantum Boltzmann case corresponds to $f = 0$ ($g = 0$) and, in the latter case, all the virial coefficients except B_1 vanish.

† We thank the referee for pointing this out to us.

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